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Quark–lepton mass relation in a realistic A_4 extension of the Standard Model

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ABSTRACT

We propose a realistic A_4 extension of the Standard Model involving a particular quark–lepton mass relation, namely that the ratio of the third family mass to the geometric mean of the first and second family masses are equal for down-type quarks and charged leptons. This relation, which is approximately renormalization group invariant, is usually regarded as arising from the Georgi–Jarlskog relations, but in the present model there is no unification group or supersymmetry. In the neutrino sector we propose a simple modification of the so-called Zee–Wolfenstein mass matrix pattern which allows an acceptable reactor angle along with a deviation of the atmospheric and solar angles from their bi-maximal values. Quark masses, mixing angles and CP violation are well described by a numerical fit.

1. Introduction

Supersymmetric Grand Unified Theories (SUSY GUTs) are very attractive from the theoretical point of view as they allow to obtain the SM from a single unified gauge group [1,2]. Apart from predicting for instance the quantization of electric charge, they reduce the number of free parameters. For example, they give the right value for the electroweak mixing angle and may provide a good framework for the understanding of the flavor problem. Indeed, several GUT models have been studied in the literature, having as prediction a mass relation between down-quark masses and the charged leptons. For instance in the $SU(5)$ unified framework Georgi and Jarlskog have found the mass relation [3]

$$m_e = \frac{1}{3} m_d, \qquad m_\mu = 3 m_s, \qquad m_\tau = m_b, \tag{1}$$

which is in good agreement with data to first approximation, assuming that holds at the GUT scale, and taking into account renormalization group running to low energies, with suitable SUSY threshold effects. Such mass relations are very welcome since, by itself, the Standard Model sheds no light on the flavor problem.

However, the Large Hadron Collider has so far not found any evidence of physics beyond the Standard Model (SM). Indeed, the only major discovery to date has been that of a new boson which is entirely consistent with the properties of the SM Higgs boson, arising from a single Higgs doublet H. In particular the LHC has not so far found any evidence for Supersymmetry (SUSY) as indicated within the simplest Grand Unified Theories (GUTs), namely those which do not involve an intermediate scale such as $SU(5)$.

Here we advocate an alternative TeV-scale approach to the flavor problem employing just the Standard Model gauge symmetry, supplemented only by a non-Abelian discrete flavor symmetry. For the latter we adopt A_4, the discrete group of even permutations of four objects isomorphic to the group of symmetries of the tetrahedron. It is the smallest group containing triplet irreducible representations. Several A_4-based flavor models have been suggested [4–6], for reviews see Refs. [7–10]. Recently three of us have proposed an $SU(3) \otimes SU(2)_L \otimes U(1)$ model [11] based on the discrete family symmetry A_4 leading to the quark–lepton mass relation:

$$\frac{m_\tau}{\sqrt{m_e m_\mu}} \approx \frac{m_b}{\sqrt{m_s m_d}}. \tag{2}$$

It is clear that Eq. (2) provides an interesting generalization of Eq. (1) which is found to be in very good agreement with data. Given that it is approximately renormalization group invariant, it holds at all mass scales [11]. In contrast to the Georgi–Jarlskog relation, Eq. (2) arises just from the flavor structure of the model, and the fact from the existence of two Higgs doublets selectively

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coupled to the up- and down-type fermions.[1] Both of these were assigned to be A_4 triplets, with one of them coupling to the down-type quarks and charged leptons. Note that the model in [11] employed an extended Higgs sector with three families of supersymmetric Higgs doublets H_u and H_d for which there is presently no evidence, with present data being consistent with a single Higgs doublet. Moreover, the model predicted $V_{ub} = 0 = V_{cb}$. While providing a good starting point for the CKM matrix, a derivation of the full quark mixing was lacking.

In the present Letter we propose an alternative discrete family symmetry A_4 model, keeping the same motivation for introducing A_4 into the Standard Model [11], namely, to shed light on the flavor problem. Indeed the model presented here provides a fully realistic description of all quark and lepton masses and mixing angles, and in particular reproduces the successful quark–lepton mass relation in Eq. (2). In contrast to the previous construction in [11] its remains closer in spirit to the SM, since we do not assume supersymmetry nor unification, keeping a single Higgs doublet H instead of multi-Higgs doublets. In particular, we assign right-handed up quarks to singlet representation of A_4 instead of triplet as in the original model. The A_4 flavor symmetry is broken by SM singlet flavons which distinguish up-type quarks from down-type quarks and charged leptons, with additional flavons in the neutrino sector, where we require extra Abelian discrete groups to distinguish these sectors. Assuming full explicit breaking of A_4 through suitable scalar flavon multiplets we show that this simple modification of the model in [11] can describe all the CKM mixing parameters. Moreover, it is straightforward to obtain also the so-called *Zee–Wolfenstein* mass matrix pattern in the neutrino sector using A_4 invariance [12]. However, since it predicts bi-maximal mixing, current neutrino oscillation data analysis [13] rule out such a pattern [14]. We propose a simple modification of the Zee–Wolfenstein model where all the mixing angle, as well as the reactor angle can be reproduced.

In the next section we introduce our model, in Section 3 we obtain our quark–lepton mass relation, in Section 4 we give the fit for the quark mixing parameters, in Section 5 we describe the neutrino mass generation mechanism and study its phenomenological implications, while in Section 6 we give our conclusions.

2. The model

The matter content and the flavor group assignment are given in Table 1. Note that all the fermions, apart of the u_R fields, are assigned to triplets of A_4.[2] In the scalar sector we have one SM Higgs doublet and four flavon fields. With respect to the model of Ref. [11] we have extra Abelian symmetries, namely Z_2^u, Z_2^d and Z_3. The reason for imposing such a symmetries is because our present model is not supersymmetric. We are replacing the SUSY-Higgs doublets H^u and H^d (triplet of A_4 in Ref. [11]) with scalar $SU_L(2)$-singlets (flavons) triplets of A_4 times the Standard Model Higgs doublet, namely

$$H^d \to H\varphi_d, \qquad H^u \to \tilde{H}\varphi_u, \tag{3}$$

where $\tilde{H} = i\sigma_2 H^*$. It is clear that the Z_2^u and Z_2^d symmetries glue the φ_u and φ_d flavons fields to the up and down-quark sectors, respectively, while the extra Z_3^ν symmetry is used to separate the charged and neutral fermion sectors.

[1] Such structure is required in supersymmetric models, but the mechanism proposed in Ref. [11] leading to Eq. (2) is more general, relying only of the two-doublet nature of the Higgs sector, as mentioned above. Here we abandon the use of supersymmetry.

[2] Therefore the present model cannot be embedded in any grand-unified framework.

The Lagrangian for quarks and charged leptons in our model is given by

$$\mathcal{L} = \frac{y^d_{\alpha\alpha'}}{M}(Q\,d_R)_\alpha H\varphi_{d_{\alpha'}} + \frac{y^l_{\alpha\alpha'}}{M}(Ll_R)_\alpha H\varphi_{d_{\alpha'}} + \frac{y^u_\beta}{M}(Q\,\varphi_u)_\beta \tilde{H} u_{R\beta'} + \text{H.c.}, \tag{4}$$

where α, α' label A_4 triplets. We remind that the product of two A_4 triplets is given by $3 \times 3 = 1 + 1' + 1'' + 3 + 3$ where the two triplet contractions can be written as the symmetric and the antisymmetric ones and denoted as 3_1, 3_2.[3] Thus we have that $\alpha = 3_1, 3_2$ while $\alpha\prime = 3$, implying that $y^{d,l}_{\alpha\alpha'}$ gives only two couplings $y_1^{d,l} \equiv y^{d,l}_{3_1 3}$ and $y_2^{d,l} \equiv y^{d,l}_{3_2 3}$. On the other hand, β and β' can be $1, 1', 1''$ in such a way that $\beta \times \beta' = 1$. Note that, while the A_4 flavor symmetry holds in the (non-renormalizable) Yukawa terms leading to charged fermion masses, we assume it to be completely broken in the scalar potential. Indeed, we assume that the scalar flavon multiplets get vacuum expectation values (vevs) in an arbitrary direction of A_4, preserving none of its subgroups. This can be easily achieved just by including terms in the scalar potential which are $SO(3)$ invariant as discussed in [15]. In this case the flavon scalar fields get vevs in arbitrary directions of A_4, that is

$$\langle\varphi_f\rangle \propto \left(v_1^f, v_2^f, v_3^f\right), \tag{5}$$

where $v_1^f \neq v_2^f \neq v_3^f$ and $f = u, d, \nu$. To complete the model we need also to specify the mechanism of neutrino mass generation, see Section 5, below.

3. The charged lepton–quark mass relation

From the A_4 contraction rules (see Appendix A) and the fact that the charged leptons and down-type quarks are in the same A_4 representations, one sees that the charged leptons and down-type quark mass matrices have the form

$$M_f = \begin{pmatrix} 0 & y_1^f v_3^f & y_2^f v_2^f \\ y_2^f v_3^f & 0 & y_1^f v_1^f \\ y_1^f v_2^f & y_2^f v_1^f & 0 \end{pmatrix}, \tag{6}$$

where $f = d, l$. This special form is the same as obtained in Refs. [11,16]. With the redefinition of variables:

$$y_1^f = a^f/v_2^f, \qquad y_2^f = b^f/v_2^f,$$
$$\alpha^f = v_3^f/v_2^f, \qquad r^f = v_1^f/v_2^f, \tag{7}$$

the mass matrix for the mass matrix in Eq. (6) takes the form

$$M_f = \begin{pmatrix} 0 & a^f\alpha^f & b^f \\ b^f\alpha^f & 0 & a^f r^f \\ a^f & b^f r^f & 0 \end{pmatrix}. \tag{8}$$

Let us now consider the system given by the following three invariants

$$\det S^f = \left(m_1^f m_2^f m_3^f\right)^2, \tag{9}$$

$$\operatorname{Tr} S^f = {m_1^f}^2 + {m_2^f}^2 + {m_3^f}^2, \tag{10}$$

$$\left(\operatorname{Tr} S^f\right)^2 - \operatorname{Tr} S^f S^f = \left({m_1^f}^2 + {m_2^f}^2\right){m_3^f}^2 + {m_1^f}^2 {m_2^f}^2, \tag{11}$$

[3] In A_4 there is only one triplet irreducible representation, here 3_1 and 3_2 are not different irreducible representations, but simply a way to indicate different contractions.

Table 1
Matter content of the model.

	L	l_R	Q	d_R	u_{R_1}	u_{R_2}	u_{R_3}	H	φ_u	φ_d	φ_ν	ξ_ν
A_4	3	3	3	3	1	$1''$	$1'$	1	3	3	3	1
Z_2^u	+	+	+	+	−	−	−	+	−	+	+	+
Z_2^d	+	−	+	−	+	+	+	+	+	−	+	+
Z_3^ν	ω	ω^2	1	1	1	1	1	1	1	1	ω	ω

where $S^f = M_f M_f^\dagger$. This system can be solved and we find

$$r^f \approx \frac{m_3^f}{\sqrt{m_1^f m_2^f}}\sqrt{\alpha^f}, \tag{12}$$

$$a^f \approx \frac{m_2^f}{m_3^f}\sqrt{\frac{m_1^f m_2^f}{\alpha^f}}, \tag{13}$$

$$b^f \approx \sqrt{\frac{m_1^f m_2^f}{\alpha^f}}, \tag{14}$$

in the limit $r^f \gg \alpha^f, 1$ and $r^f \gg b^f/a^f$. These equations are general in the sense that in the complex case, namely complex Yukawa couplings and vevs, the invariants, Eqs. (9)–(11) do not depend on the phases of the vevs v_{ui}. Indeed, the only dependence on the relative phase of the Yukawa couplings, y_1^f and y_2^2 enters in the determinant, Eq. (9), and this is negligible in the above limit. From Eqs. (12), (13), (14), one finds simple relations for the second and third family masses, namely,

$$m_2^f \approx a^f r^f, \qquad m_3^f \approx b^f r^f, \qquad \frac{m_2^f}{m_3^f} \approx \frac{a^f}{b^f}, \tag{15}$$

from which we require $a^f \ll b^f$ in order to account for the second and third family mass hierarchy. Moreover, since the charged leptons and down-type quarks couple to the same Higgs and flavons, we have[4]

$$r^l = r^d \quad \text{and} \quad \alpha^l = \alpha^d, \tag{16}$$

so that, from Eq. (12), we obtain the mass relation [11]

$$\frac{m_\tau}{\sqrt{m_e m_\mu}} \approx \frac{m_b}{\sqrt{m_s m_d}}. \tag{17}$$

Now we turn to the up-type quark sector. From the Lagrangian in Eq. (4), the up-quark mass matrix is given by

$$M_u = \begin{pmatrix} v_1^u & 0 & 0 \\ 0 & v_2^u & 0 \\ 0 & 0 & v_3^u \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \cdot \begin{pmatrix} y_1^u & 0 & 0 \\ 0 & y_{1''}^u & 0 \\ 0 & 0 & y_{1'}^u \end{pmatrix}. \tag{18}$$

In what follows we discuss the resulting structure of the quark mixing matrix.

4. Quark mixing: The CKM matrix

Recall that the down-type quark mass matrix takes the form in Eq. (6) while the up-type quark mass matrix has just been given in Eq. (18). Out of these matrices one finds the matrices $M^u \cdot M^{u\dagger}$ and $M^d \cdot M^{d\dagger}$. Their diagonalization results in two unitary matrices $V^{u,d}$ for which one can obtain approximate analytical expressions. In the down sector, from $M^d \cdot M^{d\dagger}$, one finds,

$$V_{12}^d \approx \sqrt{\frac{m_d}{m_s}}\frac{1}{\sqrt{\alpha^d}}, \tag{19}$$

$$V_{13}^d \approx \frac{m_s}{m_b^2}\sqrt{m_d m_s}\frac{1}{\sqrt{\alpha^d}}, \tag{20}$$

$$V_{23}^d \approx \frac{m_d m_s}{m_b^3}\frac{1}{\alpha^d}. \tag{21}$$

One sees that if $\alpha^d \sim \mathcal{O}(1)$ the down sector gives about the Cabibbo angle in the 1–2 plane while the mixings in the 1–3 and 2–3 planes are negligible. On the other hand, from the up-quark sector one finds, approximately

$$M_u M_u^\dagger \sim \begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix} \tag{22}$$

with coefficients of order one in front of the (i, j) if at least one of the Yukawa couplings is of order one.[5] Thus for the up-quark matrix mixing factor V^u we have that

$$V_{23}^u \approx \lambda^2, \tag{23}$$

$$V_{13}^u \approx \lambda^4, \tag{24}$$

$$V_{12}^u \approx \lambda^2, \tag{25}$$

where we have assumed

$$v_3^u : v_2^u : v_1^u = 1 : \lambda^2 : \lambda^4. \tag{26}$$

The overall quark mixing matrix is given by the product

$$V^{u\dagger} \cdot V^d.$$

One sees how the Cabibbo angle arises from the down-type quark matrix mixing factor V^d, while the V_{ub} and V_{cb} CKM mixing angles arise from the up-quark matrix mixing factor V^u. Taking $\lambda \approx 0.2$ we obtain approximately the correct value for the mixing angle. However the order-one parameters are relevant in order to exactly determinate the quark mixing angles. In order to obtain quantitative predictions for these we perform a global numerical fit. The experimental data used and the one σ error bars are given in the second column of Table 2, taking the quark masses (at the scale of the M_Z) from [18] and the quark mixing angles from [19]. The third column of Table 2 displays the values predicted by our model when the values of its parameters are those in Eqs. (27).

We note that the phases of the up couplings y_i^u can be reabsorbed by transforming the right-handed fields u_{R_i} while in the down sector not all the phases can be removed. For simplicity we

[4] Note that this relation is natural in supersymmetric models [11,17]. Here it follows from the Z_N assignments, namely the fact that the same flavon φ_d couple to down-quarks and charged leptons.

[5] We note that the magnitude of the order-one parameters that enter implicitly in Eq. (22) is relevant in order to fit the quark mass hierarchy. In particular note that the Yukawa couplings in Eq. (18) have a hierarchical structure as determined from Eq. (27).

Table 2
Experimental and predicted quark masses and mixing parameters from our fit. Quark masses (at the scale of the M_Z) have been taken from [18], while quark mixing angles have been taken from [19]. The third column displays our predicted values from Eqs. (27) which are in very good agreement with the experimental data.

Observable	Experimental value	Model prediction
m_d [MeV]	$2.9^{+0.5}_{-0.4}$	2.93
m_s [MeV]	$57.7^{+16.8}_{-15.7}$	62
m_b [MeV]	2820^{+90}_{-40}	2830
m_u [MeV]	$1.45^{+0.56}_{-0.45}$	1.63
m_c [MeV]	635 ± 86	640
m_t [GeV]	$172.1 \pm 0.6 \pm 0.9$	172.1
$\lvert V_{us}\rvert$	0.22534 ± 0.00065	0.2253
$\lvert V_{ub}\rvert$	$0.00351^{+0.00015}_{-0.00014}$	0.00347
$\lvert V_{cb}\rvert$	$0.0412^{+0.0011}_{-0.0005}$	0.0408
J	$2.96^{+0.20}_{-0.16}$	2.93

assume all the couplings to be real and we show how to make a fit of the quark mixing parameters (including the complex phase) and masses. We note that by taking ω to be the only phase in our parametrization one can fit for the CKM phase, namely the Jarlskog invariant.[6] In Table 2 we compare our theoretical predictions with the current experimental values for the quark masses and CKM mixing parameters. The theoretical predictions are for the values:

$$\begin{aligned}
&y^u_1 v^u_3 = -297\,393\ \text{MeV},\\
&y^u_{1''} v^u_3 = -15\,563\ \text{MeV},\\
&y^u_{1'} v^u_3 = 277\ \text{MeV},\\
&v^u_2/v^u_3 = 1.03\lambda^2,\\
&v^u_1/v^u_3 = 2.14\lambda^4,\\
&\alpha_d = 1.58,
\end{aligned}\tag{27}$$

where $\lambda = 0.2$.

5. Neutrino masses and mixing

As in Ref. [11] here we consider an effective way to generate neutrino masses by *à la Weinberg* by upgrading the standard dimension-five operator to the flavon case, making it dimension-six, that is[7]

$$\frac{y^\nu_\varphi}{\Lambda^2} LLHH\varphi_\nu + \frac{y^\nu_\xi}{\Lambda^2} LLHH\xi_\nu. \tag{28}$$

After electroweak symmetry breaking it gives to the following Majorana neutrino mass matrix

$$m_\nu = \begin{pmatrix} d & a & b \\ & d & c \\ & & d \end{pmatrix}, \tag{29}$$

where $a = y^\nu_\varphi/\Lambda^2 v_H^2 \langle\varphi_{\nu 3}\rangle$, $b = y^\nu_\varphi/\Lambda^2 v_H^2 \langle\varphi_{\nu 2}\rangle$, $c = y^\nu_\varphi/\Lambda^2 v_H^2 \langle\varphi_{\nu 1}\rangle$ and $d = y^\nu_\xi/\Lambda^2 v_H^2 \langle\xi_\nu\rangle$. Note that, unlike the charged fermion case, only the symmetric contractions are allowed from the first operator in Eq. (28). We remark that these parameters in the neutrino sector are unrelated to those in the charged fermion sector.

[6] However, this does not constitute a *geometrical origin* of the phase since the Yukawa couplings are complex.

[7] It is easy to formulate type-II [20–22] or inverse [23] seesaw variants of the model. However for simplicity here we keep the effective description presented above.

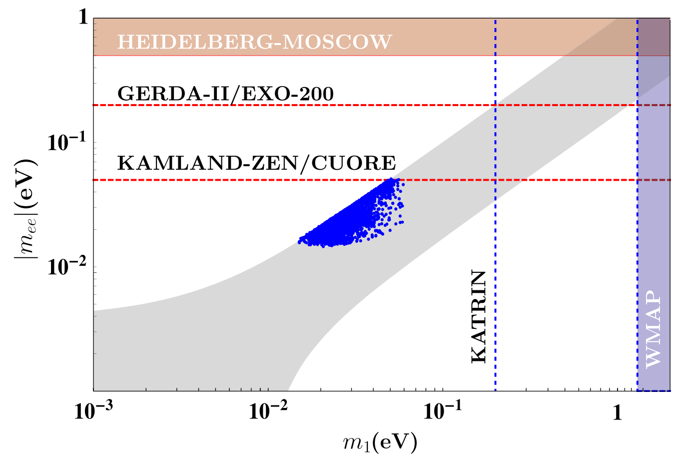

Fig. 1. Effective neutrino mass parameter m_{ee} as function of the lightest neutrino mass. The gray shaded regions correspond to the flavor-generic normal hierarchy neutrino spectra. The blue points correspond to the prediction of our A_4 model. For comparison we give the current and future sensitivities for m_{ee} [25,26] and m_ν [27,28], respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

Taking the limit where $d = 0$ the neutrino mass matrix has the well-known Zee–Wolfenstein structure, which cannot reproduce the current neutrino oscillation data [13], since it gives to bi-maximal mixing [14]. The addition of the unit matrix contribution proportional to d introduces deviations from maximal atmospheric mixing proportional to b and also introduces a non-zero reactor angle $\theta_{13} \sim (a-b)/(2d)$, while the solar angle is approximately given by $\tan 2\theta_{12} \sim 2(a+b)/d$, which reduces to maximal solar mixing in the Zee–Wolfenstein limit $d \to 0$. One finds a strict correlation between the neutrinoless double beta decay rate and the magnitude of the parameter d. In fact, as seen in Fig. 1 we find a (weak) lower bound for the neutrinoless double beta decay rate, despite the fact that the model has a normal hierarchy neutrino spectrum, this follows from the presence of the flavor symmetry.[8] In our numerical scan we also obtain a restricted set of neutrino oscillation angles. For example, the curved-shaped region in the left panel of Fig. 2 (orange in color version) corresponds to the "predicted" atmospheric angle consistent with the currently allowed values of the solar angle at 3σ, from Ref. [13]. For comparison we display also the 1σ bands for $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. In the right panel in Fig. 2 we re-express our $\sin^2\theta_{23}$ "prediction" in terms of the lightest neutrino mass m_1, again keeping undisplayed oscillation parameters at 3σ. The existence of the above restrictions reflects the fact that, as a result of the flavor symmetry, we have in total less parameters than observables to describe.

6. Conclusions

We have proposed a realistic A_4 extension of the Standard Model leading to the quark–lepton mass relation given in Eq. (2). This successful and nearly renormalization invariant mass relation generalizes the Georgi–Jarlskog formula and arises outside the context of unification. Quark masses, mixing angles and CP violation are properly accounted for, while in the neutrino sector we obtain a generalized Zee–Wolfenstein mass matrix giving an acceptable reactor angle along with a deviation of the atmospheric and solar angles from their bi-maximal values. As seen in Fig. 1 the neutrinoless double beta decay rate correlates sharply with this deviation

[8] Similar examples of A_4 models with similar features can be found in [24].

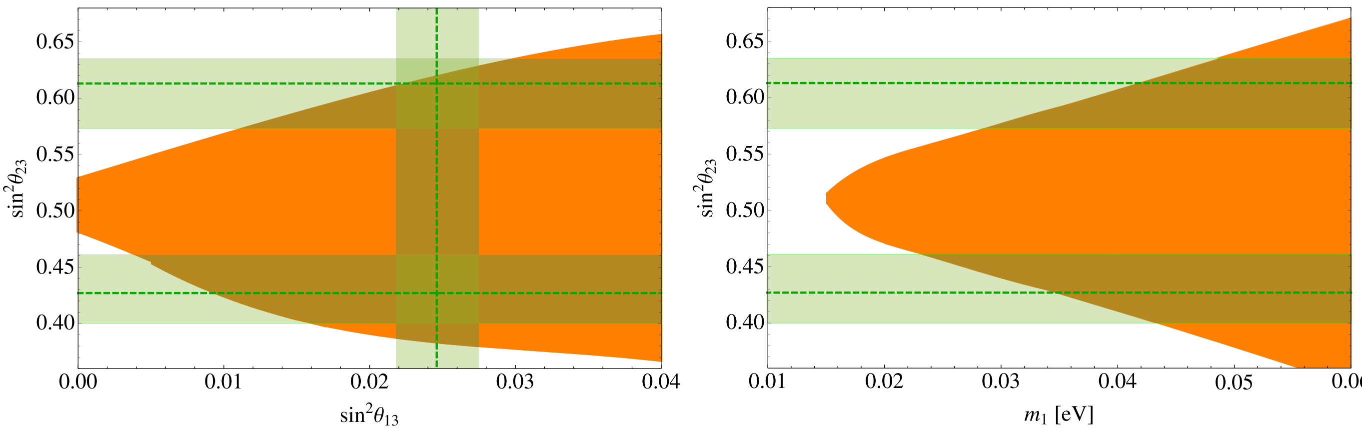

Fig. 2. Correlations between the atmospheric angle, and the reactor angle (left panel) and the lightest neutrino mass (right panel). Straight bands are the currently allowed 1σ bands of the oscillation angles, taken from Ref. [13].

parameter, with a minimum allowed value despite the fact that the model has a normal hierarchy neutrino mass spectrum. Moreover we find that the atmospheric angle correlates with the lightest neutrino mass (right panel in Fig. 2) and with the reactor angle (left panel). From the theory point of view the model treats all fermion masses effectively, as arising from non-renormalizable Yukawa-like terms.

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Appendix A. The product rules for the A_4 group

Here we adopt the $SO(3)$ basis for the generators of the A_4 group, which can be written as S and T with $S^2 = T^3 = (ST)^3 = \mathcal{I}$. A_4 has four irreducible representations, three singlets 1, 1′ and 1″ and one triplet. In the basis where S is real diagonal,

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A.1}$$

The products of singlets are:

$$\begin{aligned} &1 \otimes 1 = 1, \quad 1' \otimes 1'' = 1, \\ &1' \otimes 1' = 1'', \quad 1'' \otimes 1'' = 1' \end{aligned} \tag{A.2}$$

one has the following triplet multiplication rules,

$$\begin{aligned} (ab)_1 &= a_1 b_1 + a_2 b_2 + a_3 b_3, \\ (ab)_{1'} &= a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3, \\ (ab)_{1''} &= a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3, \\ (ab)_{3_1} &= (a_2 b_3, a_3 b_1, a_1 b_2), \\ (ab)_{3_2} &= (a_3 b_2, a_1 b_3, a_2 b_1), \end{aligned} \tag{A.3}$$

where $\omega^3 = 1$, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

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